Smooth Feedback Construction Over Spherical Polytopes

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Systems evolving on a unit sphere

- Configuration space $\mathbb{S}^2 := \{x \in \mathbb{R}^3 : x^\top x = 1\}$

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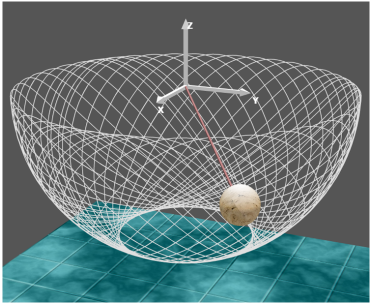
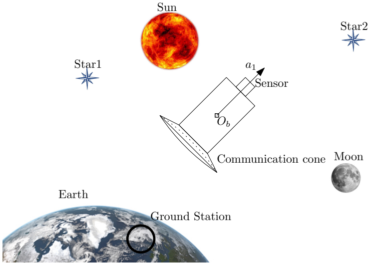


Spherical pendulum



Reduced attitude model

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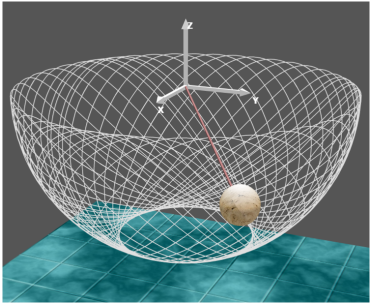

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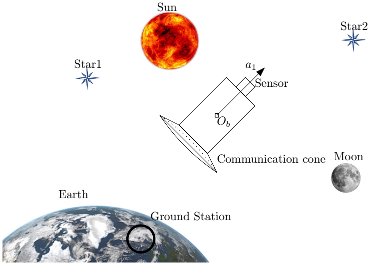

Reduced attitude model

- How to control the system state only in part of the sphere region ?

Related literature

- Control of spherical pendulum and reduced attitude model;
  - F. Bullo, R. Murray, A. Sarti et al., *Control on the sphere and reduced attitude stabilization*, in Third IFAC Symposium on Nonlinear Control Systems Design, 1995.
  - A. S. Shiriaev, H. Ludvigsen, and O. Egeland, *Swinging up the spherical pendulum via stabilization of its first integrals*, Automatica, 2004.
  - M. Ramp and E. Papadopoulos, Attitude and angular velocity tracking for a rigid body using geometric methods on the two-sphere, in 2015 European Control Conference, 2015.

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- Potential field method for full attitude model;
  - U. Lee and M. Mesbahi, *Feedback control for spacecraft reorientation under attitude constraints via convex potentials*, IEEE Transactions on Aerospace and Electronic Systems, 2014.
  - S. Kulumani and T. Lee, *Constrained geometric attitude control on SO(3)*. International Journal of Control, 2017.

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- Search-based method for full attitude model;
  - E Frazzoli, MA Dahleh, E Feron, and R Kornfeld, *A randomized attitude slew planning algorithm for autonomous spacecraft*, AIAA Guidance, Navigation, and Control Conference, 2001.
  - H. C. Kjellberg and E. G. Lightsey, *Discretized quaternion constrained attitude pathfinding*, Journal of Guidance, Control, and Dynamics, 2015.
  - X. Tan, S. Berkane, and D. V. Dimarogonas, *Constrained attitude maneuvers on SO(3): Rotation space sampling, planning and lowlevel control*, Automatica, 2020.

Related literature

- Control of spherical pendulum and reduced attitude model;
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- Control barrier function
  - G. Wu and K. Sreenath, *Safety-critical and constrained geometric control synthesis using control lyapunov and control barrier functions for systems evolving on manifolds,* American Control Conference, 2015.

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- Control of spherical pendulum and reduced attitude model;
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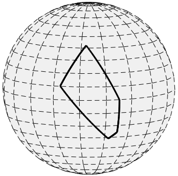
- Our contribution
  - propose a spherical polytope description and decomposition;
  - introduce the gnomonic projection that maps spherical polytopes into polytopes in $\mathbb{R}^2$;
  - the transformed dynamics is linearized to a single integrator via feedback.
  - develop a feedback control law levering with existing Euclidean navigation solutions;

Spherical polytopes

A spherical polytope P in $\mathbb{S}^2$ is a convex subset of $\mathbb{S}^2$ such that[1]

- P has only finitely many vertices;
- P is the convex hull of its vertices;
- if $x \in P$, then $-x \notin P$.



In analog, a convex polytope Q in $\mathbb{R}^2$ is a convex subset of $\mathbb{R}^2$ such that

- Q has only finitely many vertices;
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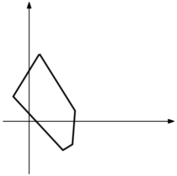

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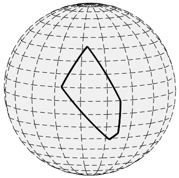

- Every P lies in a hemisphere $U_{\boldsymbol{a}} := \{\boldsymbol{x} \in \mathbb{S}^2 : \boldsymbol{a}^\top \boldsymbol{x} > 0\}$ for some vector $\boldsymbol{a} \in \mathbb{S}^2$.

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Spherical polytopes

Definition 1

A spherical polytope partitioning is a finite collection of spherical polytopes $\mathcal{P} = \{P_i\}$, $i = 1, 2, \cdots, n$, such that

1. $\mathrm{Int}(P_i) \cap \mathrm{Int}(P_j) = \emptyset$ for any distinct $P_i, P_j \in \mathcal{P}$;
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Example:

Problem formulation

The kinematic model evolving on the sphere is

$$\dot{\boldsymbol{x}} = \Pi(\boldsymbol{x})\boldsymbol{u} \tag{1}$$

where $\boldsymbol{x} \in \mathbb{S}^2$ is the state, $\boldsymbol{u} \in \mathbb{R}^m$ is the control input, $\Pi(\boldsymbol{x}) : \mathbb{R}^m \to \mathrm{T}_{\boldsymbol{x}}\mathbb{S}^2$ is a smooth matrix-valued function.

Problem 1 (Control over spherical polytopes)

Given a spherical polytope partitioning $\mathcal{P}$. Let $\mathcal{P}' \subset \mathcal{P}$, $M := \cup_i P_i, \forall P_i \in \mathcal{P}'$, and $\boldsymbol{x}_g \in M$. Assume that M is connected. Design a control input $\boldsymbol{u}$ such that

1. *all integral curves are smooth;*
2. *for all initial states $\boldsymbol{x}(0) \in M$, $\boldsymbol{x}(t) \in M$ for all $t \geq 0$;*
3. *$\boldsymbol{x}(t)$ reaches $\boldsymbol{x}_g$ asymptotically.*

Decompose-planning-control formulation

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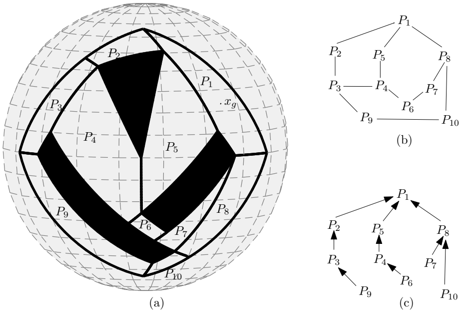

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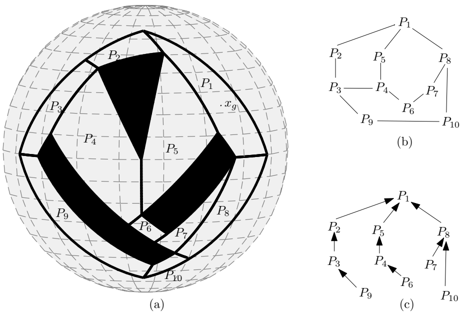
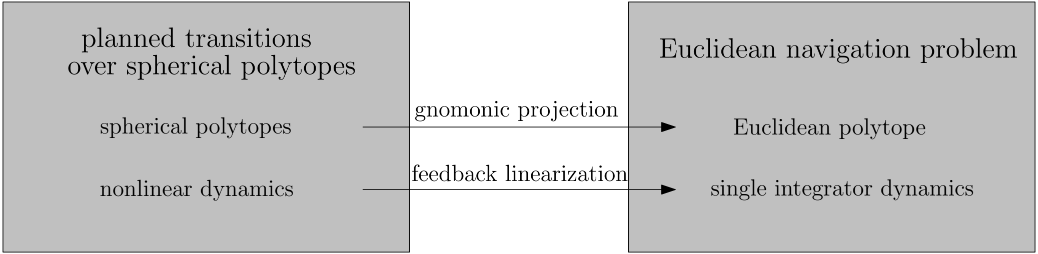


The missing stone

$\Rightarrow$ control over the planned spherical polytope transitions.

Proposed solution structure



Plenty of results exist!

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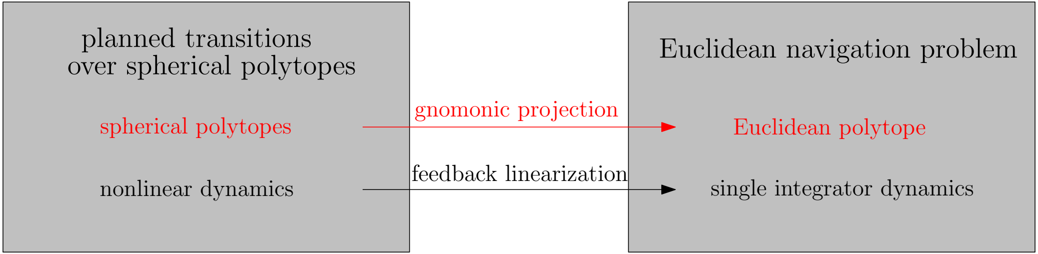

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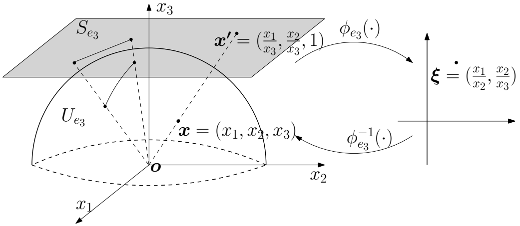
Gnomonic projection



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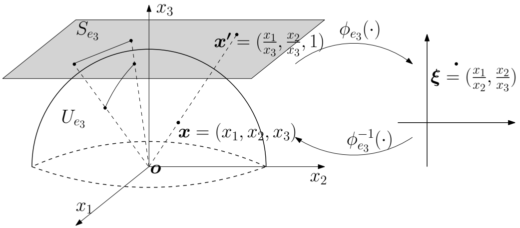


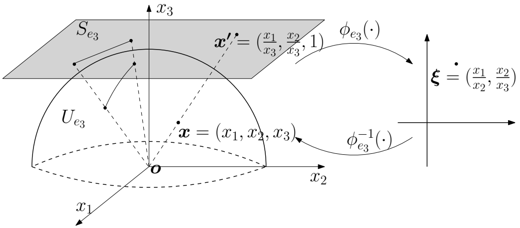
Gnomonic projection.

The gnomonic projection for $\boldsymbol{a} \in \mathbb{S}^2$, is a mapping $\phi_{\boldsymbol{a}} : \boldsymbol{x} \in U_{\boldsymbol{a}} \mapsto \boldsymbol{\xi} \in \mathbb{R}^2$

$$\phi_{\boldsymbol{a}}(\boldsymbol{x}) := J_2 R_{\boldsymbol{a}} \frac{\boldsymbol{x}}{\boldsymbol{a}^\top \boldsymbol{x}}, \tag{2}$$

where $J_2 := [I_2 \quad \mathbf{0}_{2\times 1}]$, $R_{\boldsymbol{a}}$ is a constant rotation matrix given $\boldsymbol{a}$.

Gnomonic projection



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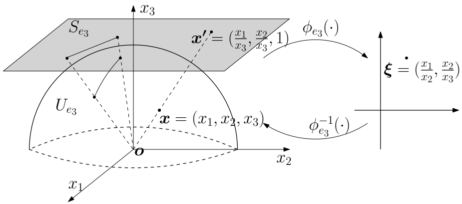
- $\phi_{\boldsymbol{a}}$ is a homeomorphism. $(U_{\boldsymbol{a}}, \phi_{\boldsymbol{a}})$ is a chart for $\mathbb{S}^2$.

Bridging spherical polytopes and Euclidean polytopes



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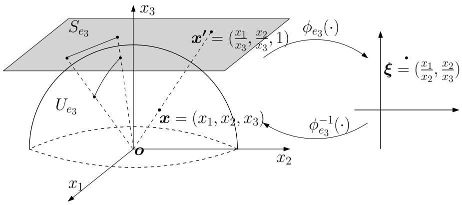

Proposition 1

The collection of charts $\mathcal{A} = \{(U_{\boldsymbol{a}}, \phi_{\boldsymbol{a}})\}_{\boldsymbol{a}\in\mathbb{S}^2}$ *is a smooth atlas for* $\mathbb{S}^2$.

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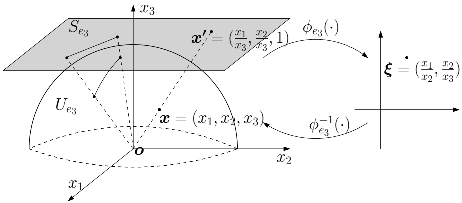

Proposition 2

For any $\mathbf{x_1}, \mathbf{x_2} \in U_{\mathbf{e_3}}$*, the gnomonic projection of* $\mathrm{seg}(\mathbf{x_1}, \mathbf{x_2})$ *forms a line segment in* $\mathbb{R}^2$*.*

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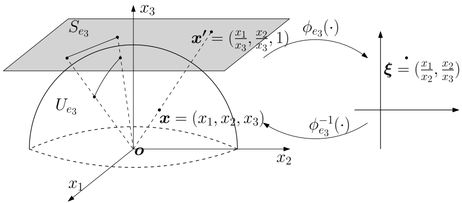

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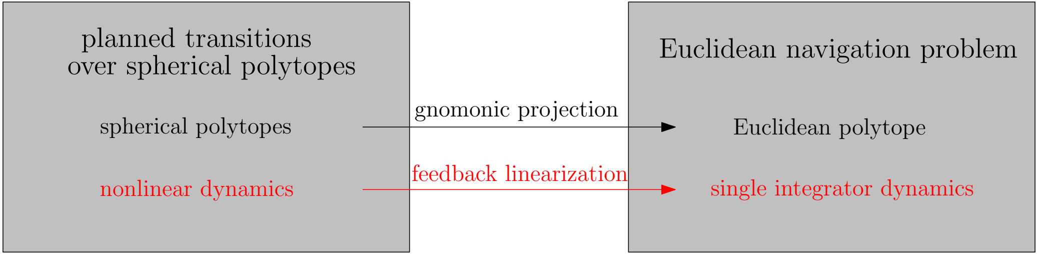
Proposition 3

Given a spherical polytope $P \subset U_{\boldsymbol{e}_3}$, the gnomonic projection of the spherical polytope is a Euclidean polytope in $\mathbb{R}^2$.

Proposed solution structure



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Projected dynamics in the Euclidean space

Recall the kinematic model is

$$\dot{\boldsymbol{x}} = \Pi(\boldsymbol{x})\boldsymbol{u}.$$

Assumption 1 (Fully actuated)

For all $\boldsymbol{x} \in \mathbb{S}^2$, $\mathrm{Im}(\Pi(\boldsymbol{x})) = \mathrm{T}_{\boldsymbol{x}}\mathbb{S}^2$.

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$$\dot{\boldsymbol{\xi}} = \nabla\phi_{\boldsymbol{a}}(\boldsymbol{x})\dot{\boldsymbol{x}} = \nabla\phi_{\boldsymbol{a}}(\boldsymbol{x})\Pi(\boldsymbol{x})\boldsymbol{u} := \Theta_{\boldsymbol{a}}(\boldsymbol{x})\boldsymbol{u} \tag{3}$$

where $\nabla\phi_{\boldsymbol{a}}(\boldsymbol{x})$ denotes the Jacobian matrix.

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Lemma 1

If Assumption 1 holds, then $\Theta_{\boldsymbol{a}}(\boldsymbol{x}) \in \mathbb{R}^{2\times m}$ *has full row rank for all* $\boldsymbol{x} \in U_{\boldsymbol{a}}$ *and all* $\boldsymbol{a} \in \mathbb{S}^2$.

Feedback linearization

Proposition 4

Consider the kinematic model (1) evolving on the 2-dimensional hemisphere $U_{\boldsymbol{a}}$ under the following feedback control law

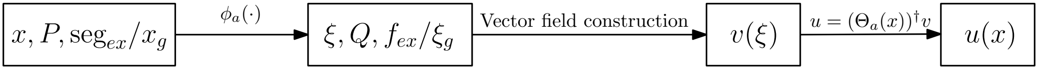
$$\boldsymbol{u} = (\Theta_{\boldsymbol{a}}(\boldsymbol{x}))^{\dagger}\boldsymbol{v} \qquad (4)$$

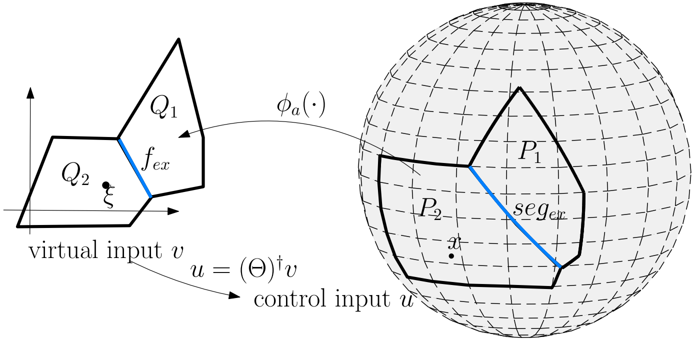
where $\boldsymbol{v} \in \mathbb{R}^2$ is a virtual control input. Then, the dynamics of the new variable $\boldsymbol{\xi} = \phi_{\boldsymbol{a}}(\boldsymbol{x})$, evolving in the Euclidean space $\mathbb{R}^2$, is

$$\dot{\boldsymbol{\xi}} = \boldsymbol{v}. \qquad (5)$$

Feedback law construction in one hemisphere

For spherical polytopes $\{P_i\}$ that jointly lie on one hemisphere $U_{\boldsymbol{a}}$,



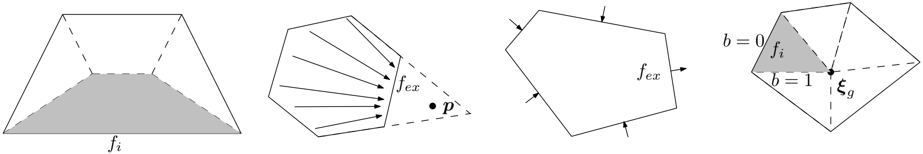


Previous results on VF construction

For a polytope Q, a vector field V is constructed by smoothly blending a cell vector field V_c and a face vector field V_{f_i}, i.e.,

$$V(\xi) = \textbf{unit}(b(\xi)V_c(\xi) + (1 - b(\xi))V_{f_i}(\xi)) \tag{6}$$

for any point ξ in the region of influence of face f_i.



region of influence of f_i — cell vector field — face vector field — blending function

- The constructed vector field is smooth on Q except for the polytope vertices.

[2] S. R. Lindemann and S. M. LaValle, International Journal of Robotics Research, 2009.

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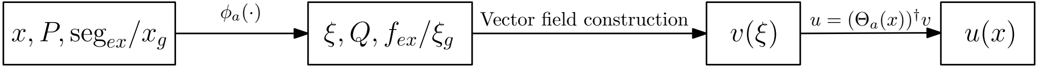

Analysis: From [2], any resulting integral curve s starting from $s(0) \in \cup_i Q_i$ is smooth, contained in $\cup_i Q_i$, $s(t)$ converges to $\boldsymbol{\xi}_g$ asymptotically.

Correspondingly, any integral curve starting from $\phi_{\boldsymbol{a}}^{-1}(s(0)) \in \cup_i P_i$ is smooth, contained in $\cup_i P_i$, and $\phi_{\boldsymbol{a}}^{-1} \circ s(t)$ converges to $\boldsymbol{x_g}$ asymptotically.

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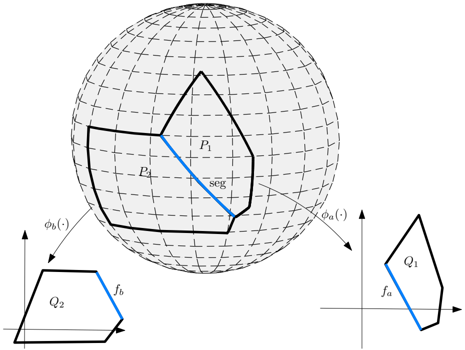
Simulation results in one hemisphere

Feedback construction across charts

A chart transition is needed when constructing a feedback law for spherical polytopes across different charts. One example:



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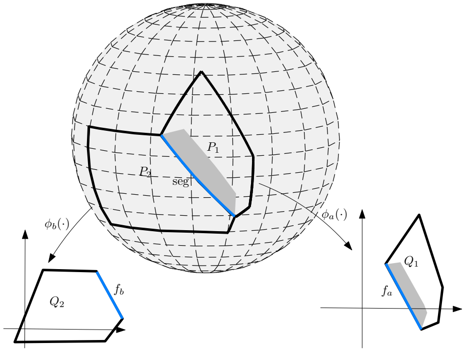


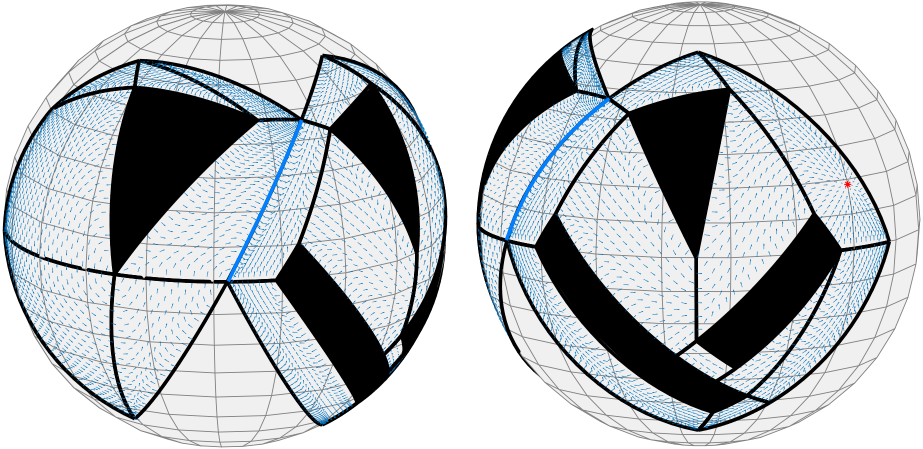
1. for $\boldsymbol{x} \in P_1 \cup P_2$ except the region of influence of $f_{\boldsymbol{a}}$ in P_1, the control law is constructed as before;

2. for $\boldsymbol{x}$ in the region of influence of $f_{\boldsymbol{a}}$ in P_1, the control input v at $\boldsymbol{\xi} = \phi_{\boldsymbol{a}}(\boldsymbol{x})$ is constructed as

$$\boldsymbol{v} = b(\xi)V_c(\xi) + (1 - b(\xi))\Theta_{\boldsymbol{a}}\Theta_{\boldsymbol{b}}^{\dagger}V_{f_{\boldsymbol{b}},2}(\xi)$$

where $V_{f_{\boldsymbol{b}},2}$ denotes the face vector field of $f_{\boldsymbol{b}}$ in Q_2. The control input $\boldsymbol{u} = \Theta_{\boldsymbol{a}}(\boldsymbol{x})^{\dagger}\boldsymbol{v}$.

Simulation results across charts



From two viewpoints

Conclusion

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  - it projects the spherical polytopes to Euclidean polytopes.
- Nonlinear kinematic dynamics can be locally transformed into a single integrator in $\mathbb{R}^2$ via feedback linearization.
- Algorithms that were originally designed for Euclidean navigation now can be used on $2-$spheres.

Future work

- Apply the spherical-polytope partitioning and gnomonic projection tools to under-actuated second-order dynamical systems evolving on higher dimensional spheres.

The End